# E. Homotopy equivalence and cobordism

**Th$^{m}$ 11:**

$\boxed{\begin{array}{l}\text{If } X \text{ and } X' \text{ are simply connected, closed 4-manifolds then} \\ \quad\text{their intersection forms are isomorphic} \\ \qquad \overset{①}{\Longleftrightarrow} \text{ (Whitehead, Milnor)} \\ \quad\text{they are homotopy equivalent} \\ \qquad \overset{②}{\Longleftrightarrow} \text{ (Wall)} \\ \quad\text{the are h-cobordant}\end{array}}$

**Proof of ①:**

($\Leftarrow$) is clear (cup product is a homotopy invariant)

($\Rightarrow$) consider the homotopy type of $X_0 = X - B^4$

note
$$H_i(X_0; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \oplus_r \mathbb{Z} & i=2 \\ 0 & i=1 \text{ or } >3 \end{cases}$$

since $\pi(X_0) = \{e\}$

Hurewitz th$^{m}$ $\Rightarrow$ $\pi_2(X_0) \cong H_2(X_0) \cong \oplus_r \mathbb{Z}$

so there is a continuous map
$$f : \underbrace{S^2 \vee \dots \vee S^2}_{r} \to M$$
that induces an isomorphism on $\pi_2$

$\therefore$ $f_* : H_*(S^2 \vee \dots \vee S^2) \to H_*(X_0)$ an isomorphism

since $S^2 \vee \dots \vee S^2$ simply connected the Whitehead theorem implies $f$ a homotopy equivalence if $X_0$ a CW complex, but handle str. on $X_0$ gives us this (this is true for top. mfds too but need to know $X_0$ has htpy type of CW complex)

now $X$ is obtained from $X_0$ by attaching a 4-cell

$\therefore$ $X$ is homotopy equivalent to $S^2 \vee \dots \vee S^2$ with a 4-cell attached by some map

$$g: S^3 \to S^2 \vee \dots \vee S^2$$

and the homotopy class of $X$ is determined by the homotopy class of $g$ i.e. an elt of $\pi_3(S^2 \vee \dots \vee S^2)$

to the homotopy class $[g]$ we can associate a symmetric matrix $A_{[g]}$ satisfying

1. $A_{[g]}$ is isomorphic to the intersection matrix of $X$
2. there is a one-to-one correspondence between symmetric $r \times r$ matricies and elements of $\pi_3(\underbrace{S^2 \vee \dots \vee S^2}_{r})$

Pontragin-Thom construction

given this the homotopy type of $X$ is determined by its intersection form!

choose a point $p_i$ on each $S^2$ in $S^2 \vee \dots \vee S^2$ (distinct from wedge pt.)

homotop $g$ to be transverse to the $p_i$

let $K_i = g^{-1}(p_i)$ <u>note:</u> by orienting the $S^2$'s and $S^3$ these are oriented links

<u>exercise:</u> each $K_i$ has a framing

<u>hint:</u> a basis $e_1, e_2$ of $T_{p_i}S^2$ gives the trivialization of $\nu(K_i)$

we now get a linking matrix for $K_1 \cup \dots \cup K_r$ which we denote $A_g$

(i.e. $m_{ij} = lk(K_i, K_j)$ for $i \neq j$
$m_{ii} =$ framing of $K_i$)

we say 2 framed links $\cup K_i$ and $\cup K_i'$ are framed cobordant

if $\exists$ a surface $\cup S_i \subset [0,1]\times S^3$ with framing such that

$\cup S_i \cap \{0\}\times S^3 = K_i$ and

$\cup S_i \cap \{0\}\times S^3 = K_i'$

and framing on $S_i$ restrict to those on $S_i$ & $S_i'$

exercise:

1) if $\cup K_i$ framed cobordant to $\cup K_i'$ then their linking matricies are isomorphic

2) if $g$ is homotopic to $g'$ then $A_g$ is isomorphic to $A_{g'}$

so there is a well-defined $A_{[g]}$

3) we can homotop $g$ so that each $K_i$ is connected

Proof of 1:

for each $K_i$ let $\Sigma_i$ be a surface in $B^4$ with boundary $K_i$

in $\hat{X} = (S^2 \vee \cdots \vee S^2) \cup B^4$, let $\overline{\Sigma}_i = \Sigma_i/_{\partial\Sigma_i}$ be a closed surface (note $g(\partial \Sigma_i) = p_i$)

exercise: the $\Sigma_i$ generate $H_2(\hat{X})$

from construction it is easy to see the intersection matrix for $\hat{X}$ is $A_g$ ✓

Proof of 2:

from above there is a well-defined map

$$\pi_3(S^2 \vee \cdots \vee S^2) \to \{\text{symmetric matrices}\}$$

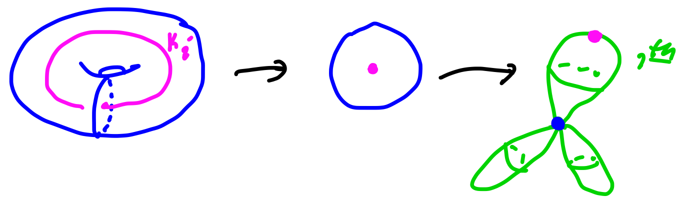
first we see this map is onto:

given $A$, let $K_1 \cup \cdots \cup K_r$ be a framed link in $S^3$ whose linking matrix is $A$

each $K_i$ has a nbhd $K_i \times D^2$ with the product structure coming from the framing

we can map

$K_i \times D^2 \to D^2 \to i^{th}\ S^2$ s.t. center point of $D^2$ goes to $p_i$ and $\partial D^2$ goes to wedge point




we now get $g_A : S^3 \to S^2 \vee \dots \vee S^2$

by sending $S^3 - \cup (K_i \times D^2)$ to wedge point

clearly $g_A^{-1}(p_i) = K_1 \cup \dots \cup K_r$ so the map is onto

now suppose $g_0, g_1 : S^3 \to S^2 \vee \dots \vee S^2$ give the same matrix A

technically they are just isomorphic, we assume they are the same and leave general case as an exercise

exercise: given the framed link $L_i$ from $g_i$ show $g_i$ can be homotoped so it came from the construction above

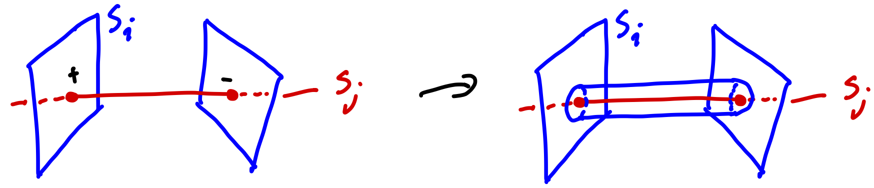
so the links $L_0$ and $L_1$ have the same linking matrix

any 2 knots in $S^3$ are cobordant so we can pair up a component of $L_0$ with a component of $L_1$ so the linking data preserved

now let $S_1, \dots S_r$ be surfaces embedded in $[0,1] \times S^3$ giving a cobordism between paired components

exercise: the $S_i$'s might intersect but their algebraic intersection is 0

so points in $S_i \cap S_j$ pair up now alter $S_i$ as follows




now $S_1 \cup \dots \cup S_r$ is embedded

since the framings on $\partial S_i$ agree we can frame $S_i$ using this framing

we can now build a map $H: S^3 \times [0,1] \to S^2 \vee \dots \vee S^2$ as above

and $H$ will be a homotopy from $g_0$ to $g_1$ (after $g_i$ modified as in exercise above)

we need a few preliminary ideas before proving part ② of the theorem

if $X$ is an oriented manifold of dimension $\geq 3$ then a spin structure on $X$ is a trivialization of $TX$ on the 1-skeleton of $X$ that extends to the 2-skeleton of $X$

exercise: you can think of an orientation of $X$ as a trivialization of $TX$ on the 0-skeleton that extends over the 1-skeleton so a spin str. is kind of a "stronger orientation"

**Remark:** if $X$ is orientable then the $1^{st}$ Steifel-Whitney class vanishes: $w_1(TX)=0$

one can show an orientable manifold

$$X \text{ is spin} \iff w_2(TX)=0$$

**Fact:** 1) if a 4-manifold $X$ is spin, then its intersection form is even

if $\pi_1(X)=\{1\}$, then $X$ is spin $\iff$ its intersection form is even

2) if $X^4$ is spin and $\sigma(X)=0$ then $X$ bounds a spin 5-manifold $W$

3) a 5D 2-handle is determined by its attaching $S^1$ and a framing: $\pi_1(SO(3))\cong \mathbb{Z}/2$

attaching a 2-handle to $B^5=h^0$ results in a $D^3$-bundle over $S^2$ (just like we saw in 4D)

with one framing you get $S^2\times D^3$

with the other you get $S^2\tilde{\times} D^3$ (the unique non-trivial bundle)

their boundaries are $S^2\times S^2$  or $S^2\tilde{\times}S^2$ 

if $\partial W$ not spin, then attaching a 2-handle with either framing gives the same thing

(the idea is you can push attaching sphere over an odd homology class to change its framing)

4) if $W$ is a spin 5-manifold and $W'=W\cup k$-handle then $W'$ is spin if $k\neq 2$ or $k=2$ and we have the "trivial" framing

## Proof of Th$^{m}$ 11 ②:

(⇐) clear

(⇒) We begin by considering $X$ a spin manifold and $X'$ homotopy equivalent to $X$ (and hence spin)

$\sigma(X \cup -X') = \sigma(X) + \sigma(-X') = \sigma(X) - \sigma(X') = 0$

so there is a spin 5-manifold $W$ s.t. $\partial W = -X' \cup X$

i.e. $W$ a cobordism from $X'$ to $X$

we need to turn $W$ into an $h$-cobordism recall this means

$\pi_1(W) = \{1\}$ and

$\imath: X \hookrightarrow W$, $\imath': X' \hookrightarrow W$ are htpy. equiv.

↪ same as $H_*(W, X) = 0$

we know $W$ has a handlebody structure:

$([0,1] \times X') \cup (1\text{-h})s \cup (2\text{-h})s \cup (3\text{-h})s \cup (4\text{-h})s$

from Section II

consider a 1-handle

It's attaching sphere can be isotoped into a ball in $\{1\} \times X'$

boundary sum

exercise: $([0,1] \times X') \cup (1\text{-h}) \cong ([0,1] \times X') \natural S^1 \times D^4$

so the upper boundary of $([0,1] \times X') \cup (1\text{-h})$ is $X' \# S^1 \times S^3$

if we attach a 3-handle to $[0,1] \times X'$ we also get a cobordism with upper boundary $X' \# S^1 \times S^3$

i.e. $([0,1] \times X') \cup 3\text{-h} \cong ([0,1] \times X') \natural S^3 \times D^2$

∴ we can replace all 1-handles with 3-handles to get another cobordism from $X'$ to $X$ (still call it $W$)

we can similarly replace 4-handles with 2-handles
(just turn $W$ upside down)

so we can assume $W$ is

$$([0,1]\times X') \cup (2\text{-}h)s \cup (3\text{-}h)s$$

since $X'$ is simply connected we can assume all
the attaching spheres for 2-handles are unknots
in balls in $\{1\}\times X'$
and since $W$ spin, framings are trivial so

$$\partial_+ W_2 = X' \#_k S^2\times S^2 \quad \text{where } k = \#\text{ 2-handles}$$

$W_2 = ([0,1]\times X') \cup$ 2-handles

of course, upside down we see

$$\partial_+ \widetilde{W}_2 = X \#_\ell S^2\times S^2 \text{ where } \ell = \#\text{ 3-handles}$$

since $\partial_+(\overline{W}_2) \cong \partial_+ W_2$ we see $k=\ell$

**Thm 12 (Wall):**

$$\boxed{\text{if } X, X' \text{ are homotopy equivalent, simply connected 4-manifolds, then there is some } k \text{ such that } X\#_k S^2\times S^2 \text{ is diffeomorphic to } X'\#_k S^2\times S^2}$$

**Proof:** we just proved this for $X$ spin, the general case will be shown below

**Major Open Question:** in theorem above can you always take $k=1$
(you can in all known examples)

let $W = W' \cup W''$ where $W' = W_2$ and $W'' = \overline{W - W_2}$ closure
(i.e. we cut $W$ along $X' \#_k S^2 \times S^2$)

so there is a diffeomorphism $f : X \#_k S^2 \times S^2 \to X' \#_k S^2 \times S^2$
such that $W = W' \cup_f W''$

we would like to find a diffeo. $h : X' \#_k S^2 \times S^2 \to X' \#_k S^2 \times S^2$
such that in $W' \cup_{h \circ f} W''$ the 3-handles
homologically cancel the 2-handles

if we can do this, then $\hat{W} = W' \cup_{g \circ f} W''$ will
be simply connected and $H_*(\hat{W}, X') = 0$
so $\hat{W}$ is the desired h-cobordism!

for this we need

**Th$^m$ 13 (Wall):**

> let $X$ be a smooth, simply connected, closed
> 4-manifold with an indefinite intersection form
> Then any automorphism of the intersection form
> of $X \# S^2 \times S^2$ can be realized by a diffeomorphism

now we can add an extra $S^2 \times S^2$ to $\partial_+ W_2$ if needed to apply
this theorem by adding a cancellin 2/3-pair

since the cocores $c_1, \dots c_k$ of the 2-handles can form part of
a basis for the $S^2 \times S^2$ part of $H_2(X' \#_k S^2 \times S^2)$
let $c_1', \dots c_k'$ be the elements (i.e. $c_i \cdot c_j' = \delta_{ij}$)

note: we need the attaching spheres of 3-handles to
to homologically map to $c_i'$ to kill homology

similarly let $a_1, \dots, a_k$ be the attaching spheres of the 3-handles

there is an automorphism

$$\psi : H_2(\#_k S^2 \times S^2) \to H_2(\#_k S^2 \times S^2)$$

taking the $a_i$ to $c_i'$

let $\phi : H_2(X) \to H_2(X')$ be the isomorphism induced by the homotopy equivalence

we would like a diffeo. $h : X' \#_k S^2 \times S^2 \to X' \#_k S^2 \times S^2$

st. $(h \circ f)_* = \phi \oplus \psi$

to this end consider $(\phi \oplus \psi) \circ f_*^{-1}$

this is an automorphism of $H_2(X' \#_k S^2 \times S^2)$ so by Th$^{\underline{m}}$ 13

$\exists$ an $h$ realizing it and we are done!

now if $X, X'$ are not spin the we are still done if $\partial W_2 = X' \#_k S^2 \times S^2$

but from above we know it must be $X' \#_{k_1} S^2 \times S^2 \#_{k_2} S^2 \tilde{\times} S^2$

and since $X'$ is not-spin this is diffeo to $X' \#_{k_1 + k_2} S^2 \times S^2$

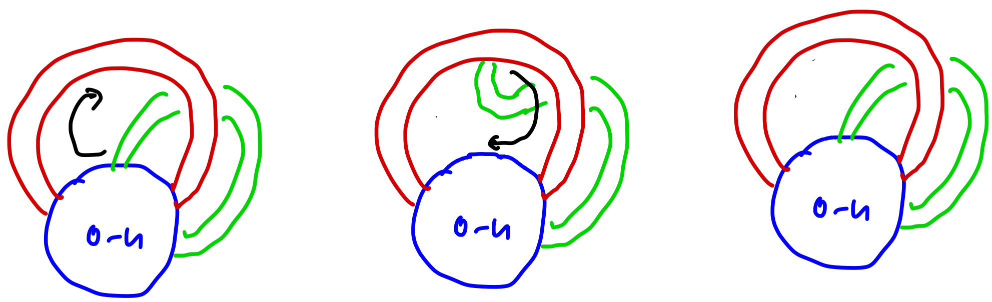
and we are done $\blacksquare$

**Proof of Th$^{\underline{m}}$ 13:**

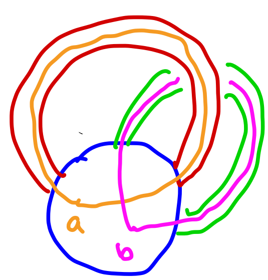
let's begin by constructing some diffeomorphisms of 4-manifolds we start by assuming X has no 1 or 3-handles

recall if we isotop the attaching regions of the 2-handles we get diffeomorphic manifolds but if we isotop untill the handles return to their original position, then we get a diffeomorphism of a fixed manifold

<u>recall in 2D</u>




on homology we see

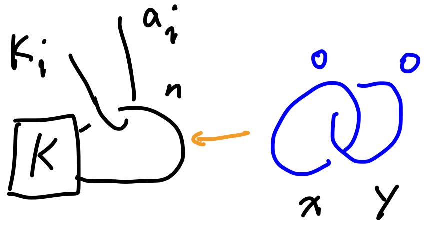



$a \mapsto a$
$b \mapsto b+a$

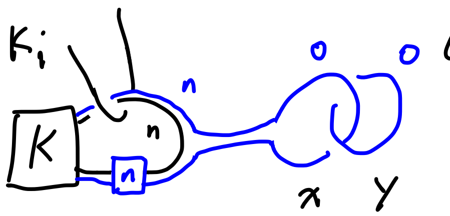
so this is a Dehn twist!

we now consider the 4D case

homology class of $K$ is $a$
$K_1$ is $a_i$

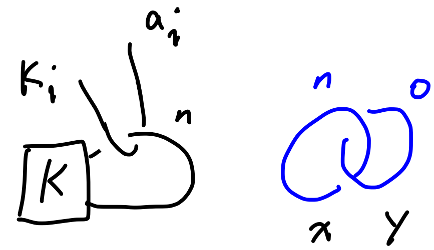



slide




$x \mapsto x+a$
all others fixed

lemma 10 shows how to slide $K, K_i$ over $y$ to unlink blue




algebraically $K_i$ slides over $y$ $lk(K,K_1)$ times to unlink and $K$ slides over $y$ $n$ times

if $n$ is even then we can slide the left blue over $y$, $-\frac{n}{2}$ time to get to 0 framing and back to the original Kirby diagram $\therefore$ we get a diffeo. of $X$ and on homology

$$x \longmapsto x + a - \tfrac{1}{2} n$$
$$y \longmapsto y$$
$$a \longmapsto a - ny$$
$$a_1 \longmapsto a_1 - lk(K,K_1)y$$

if $n$ is odd then we can go through the above isotopy again, but now the framing on "$x$" is $n+1$ after the slide so we will get back to the original diagram

now the diffeo. acts on homology by

$$x \longmapsto x + a - n$$
$$y \longmapsto y$$
$$a \longmapsto a - 2n$$
$$a_1 \longmapsto a_1 - 2\,lk(K,K_1)y$$

intersection form

Wall proved that any automorphism of $(H_2(X), Q_X)$ is a composition of such maps $\therefore \exists$ a diffeo inducing it now if X has 1 or 3-handles, we can perform handle slides (like we did in the h-cobordism thm) to do column operations on the maps

$$C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \qquad \text{(CW chain groups)}$$

untill they are of the form $I$ and $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ respectively

then do above on classes representing elts in $\ker \partial_2 / \operatorname{im} \partial_3$ $\blacksquare$